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The zrank conjecture and restricted Cauchy matrices

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Abstract

The rank of a skew partition λ/μ , denoted $\text{rank}(\lambda/\mu)$, is the smallest number r such that λ/μ is a disjoint union of r border strips. Let $s_{\lambda/\mu}(1^t)$ denote the skew Schur function $s_{\lambda/\mu}$ evaluated at $x_1 = \cdots = x_t = 1$, $x_i = 0$ for $i > t$. The zrank of λ/μ , denoted $\text{zrank}(\lambda/\mu)$, is the exponent of the largest power of t dividing $s_{\lambda/\mu}(1^t)$. Stanley conjectured that $\text{rank}(\lambda/\mu) = \text{zrank}(\lambda/\mu)$. We show the equivalence between the validity of the zrank conjecture and the nonsingularity of restricted Cauchy matrices. In support of Stanley's conjecture we give affirmative answers for some special cases.

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1. Introduction

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of an integer n , i.e., $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ and $\lambda_1 + \lambda_2 + \dots = n$. The number of positive parts of λ is called the length of λ , denoted $\ell(\lambda)$. The *Young diagram* of λ may be defined as the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$ and $1 \leq i \leq \ell(\lambda)$. A Young diagram can also be represented in the plane by an array of squares justified from the top and left corner with $\ell(\lambda)$ rows and λ_i squares in row i . A square (i, j) in the diagram is the square in row i from the top and column j from the left. The content of (i, j) , denoted $\tau((i, j))$, is given by $j - i$. The *rank* of λ , denoted $\text{rank}(\lambda)$, is the length of the main diagonal of the diagram of λ . Given two partitions λ and μ , we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i . If $\mu \subseteq \lambda$, we define a *skew partition* λ/μ , whose Young diagram is obtained from the Young diagram of λ by peeling off the Young diagram of μ from the upper left corner.

We assume that the reader is familiar with the notation and terminology on symmetric functions in [9]. In connection with tensor products of Yangian modules, Nazarov and Tarasov [8] give a generalization of a rank to a skew partition λ/μ . Recently Stanley developed a general theory of minimal border strip decompositions and gave several simple equivalent characterizations of $\text{rank}(\lambda/\mu)$ in [10]. One of the characterizations of the rank of a skew partition λ/μ says that $\text{rank}(\lambda/\mu)$ is the smallest integer r such that the Young diagram of λ/μ is the disjoint union of r border strips. Let $s_{\lambda/\mu}(1^t)$ denote the skew Schur function $s_{\lambda/\mu}$ evaluated at $x_1 = \dots = x_t = 1$, $x_i = 0$ for $i > t$. The *zrank* of λ/μ , denoted $\text{zrank}(\lambda/\mu)$, is the largest power of t dividing the polynomial $s_{\lambda/\mu}(1^t)$. Stanley conjectured that the equality $\text{rank}(\lambda/\mu) = \text{zrank}(\lambda/\mu)$ always holds, which we call the *zrank conjecture*.

In his combinatorial approach to the zrank conjecture in [10], Stanley defined the snake sequence and the interval sets for a skew partition λ/μ . In Section 2 for each interval set \mathcal{J} of λ/μ we define an interval permutation $\sigma_{\mathcal{J}}$. Let $\text{cr}(\mathcal{J})$ be the number of crossings of \mathcal{J} , and let $\text{inv}(\sigma_{\mathcal{J}})$ be the number of inversions of $\sigma_{\mathcal{J}}$. We show that $\text{cr}(\mathcal{J})$ and $\text{inv}(\sigma_{\mathcal{J}})$ have the same parity.

Stanley generalized the code of a partition to the code of a skew partition, and obtained a two-line binary sequence in [10]. This sequence is called the *partition sequence* by Bessenrodt [1,2]. Given a minimal border strip decomposition \mathbf{D} of λ/μ , let $P_{\mathbf{D}}$ be the set of the contents of the lower left-hand squares of the border strips in \mathbf{D} , and let $Q_{\mathbf{D}}$ be the set of the contents of the upper right-hand squares. Using the partition sequence, we show that $P_{\mathbf{D}}$ and $Q_{\mathbf{D}}$ are uniquely determined by the shape of the skew partition λ/μ in Section 3, i.e., these two sets are independent of the minimal border strip decomposition \mathbf{D} . For a given skew partition, we find a connection between the values of these two sets and the paired integers of the interval set.

Outside decompositions are introduced by Hamel and Goulden [6] and are used to give a unified approach to the determinantal expressions for the skew Schur functions including the Jacobi–Trudi determinant, its dual, the Giambelli determinant

and the ribbon determinant. For any outside decomposition, Hamel and Goulden derive a determinantal formula with ribbon Schur functions as entries. Their proof is based on a lattice path construction and the Gessel–Viennot methodology [4,5]. In Section 4 we employ the determinantal formula in the case of the greedy border strip decomposition and give the evaluation of $(t^{-\text{rank}(\lambda/\mu)} s_{\lambda/\mu}(1^t))_{t=0}$. As a consequence we obtain the combinatorial description of $(t^{-\text{rank}(\lambda/\mu)} s_{\lambda/\mu}(1^t))_{t=0}$ in terms of the interval sets of λ/μ given by Stanley [10, Eq. (30)].

Based on the above results, we give an equivalent characterization of the zrank conjecture. Given two positive integer sequences, we define a *restricted Cauchy matrix* corresponding to these two sequences. The main objective of this paper is to show that the zrank conjecture holds for any skew partition if and only if all the restricted Cauchy matrices are nonsingular. We present a constructive proof for this equivalence in Section 5. Using some fundamental properties of determinants, we confirm the nonsingularity of the restricted Cauchy matrices for several special classes of skew partitions.

2. Snake sequences and interval sets

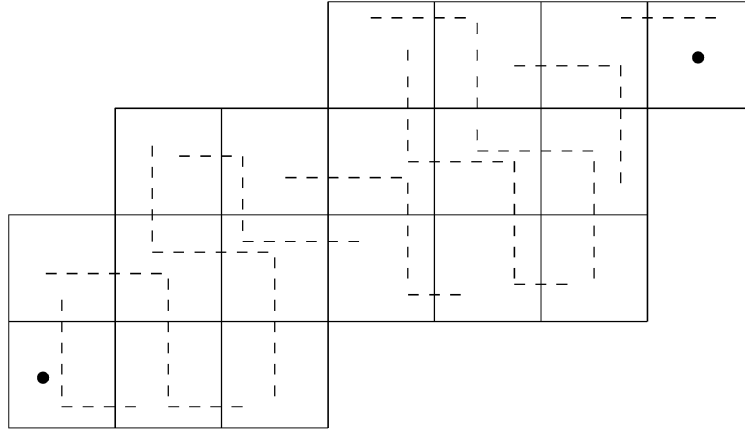
We follow the terminology of Stanley [10] on snake sequences and interval sets, which are helpful notions for the enumeration of the minimal border strip decompositions of a skew partition λ/μ . Let us consider the bottom-right boundary lattice path with steps (0,1) or (1,0) from the bottom-leftmost point of the diagram of λ/μ to the top-rightmost point. We regard this path as a sequence of edges e_1, e_2, \dots, e_k . For an edge e in this path we define a subset S_e of squares of λ/μ , called a *snake*. If there exists no square having e as an edge, then we have the set $S_e = \emptyset$. Let (i, j) be the unique square of λ/μ having e as an edge. If e is horizontal, then we define

$$S_e = \lambda/\mu \cap \{(i, j), (i-1, j), (i-1, j-1), (i-2, j-1), (i-2, j-2), \dots\}. \quad (1)$$

If e is vertical, we then define

$$S_e = \lambda/\mu \cap \{(i, j), (i, j-1), (i-1, j-1), (i-1, j-2), (i-2, j-2), \dots\}. \quad (2)$$

For example, the nonempty snakes of the skew shape $(7, 6, 6, 3)/(3, 1)$ are shown in Fig. 1, and the two snakes with just one square are shown with a single bullet. The *length* $\ell(S)$ of a snake S is defined to be one less than its number of squares. For an empty snake S , let $\ell(S) = -1$. A *right snake* is a snake of even length and of the form (1), and a *left snake* is a snake of even length and of the form (2). From the boundary lattice path we obtain a sequence of snakes: $(S_{e_1}, S_{e_2}, \dots, S_{e_k})$. The *snake sequence* of λ/μ , denoted $\text{SS}(\lambda/\mu)$, is defined by replacing a left snake of length $2m$

Fig. 1. Snakes of the skew partition $(7, 6, 6, 3)/(3, 1)$.

with the symbol L_m in the sequence $(S_{e_1}, S_{e_2}, \dots, S_{e_k})$, replacing a right snake of length $2m$ with R_m , and replacing a snake of odd length with O . From Fig. 1, we see that

$$\text{SS}((7, 6, 6, 3)/(3, 1)) = L_0 L_1 O O O O L_2 R_2 R_1 O R_0.$$

Let $\text{rank}(\lambda/\mu) = r$, and let $\text{SS}(\lambda/\mu) = q_1 q_2 \cdots q_k$. An *interval set* \mathcal{J} of λ/μ is defined to be a collection of r ordered pairs $\{(u_1, v_1), (u_2, v_2), \dots, (u_r, v_r)\}$ such that

1. $u_i \neq u_j$ and $v_i \neq v_j$ for $1 \leq i < j \leq r$.
2. $1 \leq u_i < v_i \leq k$ and $u_i \neq v_j$ for $1 \leq i, j \leq r$.
3. $q_{u_i} = L_s$ and $q_{v_i} = R_{s'}$ for some s and s' (depending on i).

Let $\text{cr}(\mathcal{J})$ denote the number of crossings of \mathcal{J} , i.e., the number of pairs (i, j) for which $u_i < u_j < v_i < v_j$. According to [10, Proposition 4.3], there exists a unique interval set $\mathcal{J}_0 = \{(w_1, y_1), (w_2, y_2), \dots, (w_r, y_r)\}$ such that $\text{cr}(\mathcal{J}_0) = 0$. From [10], we see that $\text{SS}(\lambda/\mu)$ has exactly r left snakes and r right snakes. For an interval set $\mathcal{J} = \{(u_1, v_1), (u_2, v_2), \dots, (u_r, v_r)\}$, we may impose a linear order $u_1 < u_2 < \cdots < u_r$ on its elements. Then there exists a unique permutation σ relative to \mathcal{J}_0 such that for each i

$$u_i = w_i \quad \text{and} \quad v_i = y_{\sigma_i}. \quad (3)$$

Thus, each interval set \mathcal{J} is associated to a permutation σ_I , which we call the *interval permutation* of \mathcal{J} with respect to \mathcal{J}_0 . Given a permutation σ , let $\text{inv}(\sigma)$ denote the number of inversions of σ , i.e., the number of pairs (i, j) satisfying $i < j$ but $\sigma_i > \sigma_j$.

Proposition 2.1. Given a skew partition λ/μ and an interval set \mathcal{J} of λ/μ , let $\sigma_{\mathcal{J}}$ be the interval permutation with respect to \mathcal{J}_0 . Then we have

$$\text{cr}(\mathcal{J}) \equiv \text{inv}(\sigma_{\mathcal{J}}) \pmod{2}. \quad (4)$$

Proof. First we give a geometric representation of $\text{cr}(\mathcal{J})$. For each interval (u_i, v_i) of \mathcal{J} we draw an arc on top of $\text{SS}(\lambda/\mu)$ which connects two snakes q_{u_i} and q_{v_i} . For a given pair (i, j) with $i < j$, the two arcs (u_i, v_i) and (u_j, v_j) are said to be noncrossing if $u_i < u_j < v_j < v_i$. In this terminology $\text{cr}(\mathcal{J})$ equals the number of crossings.

To determine the inversions of $\sigma_{\mathcal{J}}$, we replace q_{w_i} by F_i and q_{v_i} by G_i in $\text{SS}(\lambda/\mu)$ for each i . Clearly, $\sigma_{\mathcal{J}}$ is a bijection from $\{F_1, F_2, \dots, F_r\}$ to $\{G_1, G_2, \dots, G_r\}$. We now represent the snakes of $\text{SS}(\lambda/\mu)$ with respect to the order $F_1, F_2, \dots, F_r, G_r, G_{r-1}, \dots, G_1$ by moving G_1 to the right of the rightmost element if G_1 itself is not the rightmost element, and repeating this process until we achieve the desired order. It follows that $\text{inv}(\sigma_{\mathcal{J}})$ equals the number of crossings in the above representation. Note that at each step of moving G_i to the proper position, the number of crossings in the diagram can only change by an even number. This completes the proof. \square

For example, let $\lambda/\mu = (8, 8, 7, 4)/(4, 1, 1)$. Fig. 2 shows the snake sequence $\text{SS}((8, 8, 7, 4)/(4, 1, 1))$, from which we see that

$$\mathcal{J}_0 = \{(1, 12), (3, 11), (4, 5), (8, 9)\}.$$

Let us illustrate the proof of Proposition 2.1 by the example $\mathcal{J} = \{(1, 9), (3, 12), (4, 5), (8, 11)\}$, for which we have $\sigma_{\mathcal{J}} = [4, 1, 3, 2]$. The crossings of \mathcal{J} are shown in Fig. 3, where we relabel the snakes as described in the proof. Fig. 4 demonstrates the diagram after moving G_3 which has two more crossings. It is evident that

$$\text{cr}(\mathcal{J}) = 2, \quad \text{inv}(\pi) = 4, \quad \text{cr}(\mathcal{J}) \equiv \text{inv}(\pi) \pmod{2}. \quad (5)$$

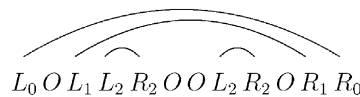


Fig. 2. Parenthesization of the snake sequence $\text{SS}((8, 8, 7, 4)/(4, 1, 1))$.

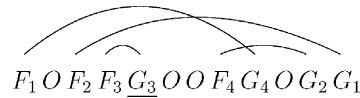


Fig. 3. Before moving G_3 .



Fig. 4. After moving G_3 .

3. Minimal border strip decompositions

We recall the notion of the *reduced code* of a skew partition λ/μ , denoted $c(\lambda/\mu)$. The reduced code $c(\lambda/\mu)$ is also known as the *partition sequence* of λ/μ [1,2]. Consider the two boundary lattice paths of the diagram of λ/μ with steps (0, 1) or (1, 0) from the bottom-leftmost point to the top-rightmost point. Replacing each step (0, 1) by 1 and each step (1, 0) by 0, we obtain two binary sequences by reading the lattice paths from the bottom-left corner to the top-right corner. Denote the top-left binary sequence by f_1, f_2, \dots, f_k , and the bottom-right binary sequence by g_1, g_2, \dots, g_k . The reduced code $c(\lambda/\mu)$ is defined by the two-line array

$$\begin{array}{cccc} f_1 & f_2 & \cdots & f_k \\ g_1 & g_2 & \cdots & g_k \end{array}.$$

The reduced code of the skew partition $(5, 4, 3, 2)/(2, 1, 1)$ in Fig. 5 is

$$\begin{array}{cccccccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}.$$

A *diagonal* with content j of λ/μ , denoted $d_j(\lambda/\mu)$, is the set of all the squares in λ/μ having content j . Suppose that the length of $c(\lambda/\mu)$ is k . It is obvious that λ/μ has $k - 1$ diagonals. Let ϵ be the smallest content of λ/μ . For each $i : 1 \leq i \leq k - 1$, we put the diagonal $d_{\epsilon+i-1}$ between the i th column and $(i + 1)$ th column of $c(\lambda/\mu)$. Then we obtain a connection between the diagonals of λ/μ and the reduced code $c(\lambda/\mu)$.

Recall that a skew partition λ/μ is said to be *connected* if the interior of the Young diagram of λ/μ is a connected set. A *border strip* is a connected skew partition with no 2×2 square. Define the size of a border strip B as the number of squares of B , and define the *height* $ht(B)$ of B as one less than its number of rows. We say

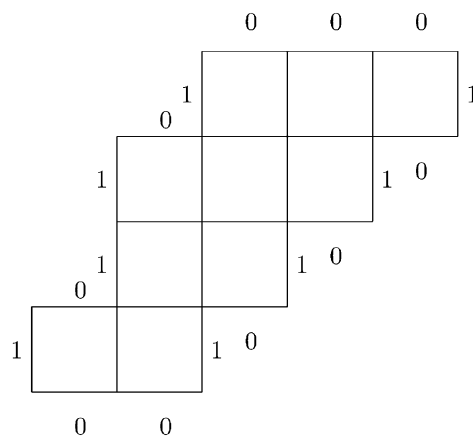


Fig. 5. Constructing the reduced code of $(5, 4, 3, 2)/(2, 1, 1)$.

that $B \subset \lambda/\mu$ is a border strip of λ/μ if $\lambda/\mu - B$ is a skew partition ν/μ . A border strip B of λ/μ is said to be *maximal* if there does not exist another border strip $B' \subset \lambda/\mu$ such that $B \subset B'$. A *border strip decomposition* [9] of λ/μ is a partition of the squares of λ/μ into pairwise disjoint border strips. A *greedy border strip decomposition* of λ/μ is obtained by successively removing the maximal border strip from λ/μ . A border strip decomposition is *minimal* if there does not exist a border strip decomposition with a fewer number of border strips.

Stanley [10, Proposition 2.2] has shown that the rank of a skew partition λ/μ is equal to the number of border strips in a minimal border strip decomposition of λ/μ , and it is also equal to the number of $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$ columns of $c(\lambda/\mu)$. As a consequence, a greedy border strip decomposition is minimal, because when we successively remove the maximal border strips from λ/μ a column $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$ of $c(\lambda/\mu)$ changes into $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ and a column $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$ changes into $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$.

Suppose that $\text{rank}(\lambda/\mu) = r$. Given a minimal border strip decomposition $\mathbf{D} = \{B_1, B_2, \dots, B_r\}$ of λ/μ , let

$$P_{\mathbf{D}} = \{\tau(\text{init}(B_1)), \tau(\text{init}(B_2)), \dots, \tau(\text{init}(B_r))\}$$

and

$$Q_{\mathbf{D}} = \{\tau(\text{fin}(B_1)), \tau(\text{fin}(B_2)), \dots, \tau(\text{fin}(B_r))\},$$

where $\text{init}(B_i)$ is the lower left-hand square of B_i and $\text{fin}(B_i)$ is the upper right-hand square. The following proposition shows that $P_{\mathbf{D}}$ and $Q_{\mathbf{D}}$ are independent of the minimal border strip decomposition \mathbf{D} .

Proposition 3.1. *Let $\mathcal{J}_0 = \{(w_1, y_1), (w_2, y_2), \dots, (w_r, y_r)\}$ be the interval set of λ/μ with $\text{cr}(\mathcal{J}_0) = 0$. Let ϵ be the smallest value among the contents of the squares of λ/μ . Let \mathbf{D} be a minimal border strip decomposition of λ/μ . Then we have*

$$P_{\mathbf{D}} = \{\epsilon + w_i - 1 | 1 \leq i \leq r\} \quad \text{and} \quad Q_{\mathbf{D}} = \{\epsilon + y_i - 2 | 1 \leq i \leq r\}. \quad (6)$$

Proof. By [10, Proposition 2.1], we see that the operation of removing a border strip B of size p from λ/μ corresponds to the operation of choosing i with the i th column being $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$ and the $(i + p)$ th column being $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$, and then replacing the i th column with $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ and the $(i + p)$ th column with $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$. Moreover, the lower left-hand square of B lies on the diagonal d_i , and the upper right-hand square of B lies on the diagonal d_{i+p-1} . Therefore

$$\tau(\text{init}(B)) = \epsilon + i - 1 \quad \text{and} \quad \tau(\text{fin}(B)) = \epsilon + i + p - 2.$$

It follows that $P_{\mathbf{D}}$ and $Q_{\mathbf{D}}$ are determined by the indices of the columns $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$ and $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$ of $c(\lambda/\mu)$ respectively. Since $\{w_i\}$ is the set of indices of columns $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$ of $c(\lambda/\mu)$, and $\{y_i\}$ is the set of indices of $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$, we get the desired assertion. \square

4. Giambelli-type determinantal formulas

In this section, we obtain a determinantal formula for the quantity given by Stanley based on the Giambelli-type formula for skew Schur functions. Let λ/μ be a skew diagram. A border strip decomposition of λ/μ is said to be an *outside decomposition* if every strip in the decomposition has an initial square on the left or bottom perimeter of the diagram and a terminal square on the right or top perimeter, see Fig. 6. It is obvious that a greedy border strip decomposition of λ/μ is an outside decomposition.

The notion of the cutting strip of an outside decomposition is introduced by Chen et al. [3], which is used to give a transformation theorem on the Giambelli-type determinantal formulas for the skew Schur function.

We proceed to construct a cutting strip for an edgewise connected skew partition λ/μ . Suppose that λ/μ has k diagonals. The cutting strip of an outside decomposition is defined to be a border strip of length k . Given an outside decomposition, we may assign a direction to each square in the diagram. Starting with the bottom-left corner of a strip, we say that a square of a strip has up direction (respectively right direction) if the next square in the strip lies on its top (respectively to its right). Notice that the strips in any outside decomposition of λ/μ are nested in the sense that the squares in the same diagonal of λ/μ all have up direction or all have right direction. Based on this property, the cutting strip ϕ of an outside decomposition \mathbf{D} of λ/μ is defined as follows: for $i = 1, 2, \dots, k - 1$ the i th square in ϕ keeps the same direction as the i th diagonal of λ/μ with respect to \mathbf{D} . For any two integers p, q a strip $[p, q]$ is defined by the following rule: if $p \leq q$, then let $[p, q]$ be the segment of ϕ from the square with content p to the square with content q ; if $p = q + 1$, then let $[p, q]$ be the empty strip; if $p > q + 1$, then $[p, q]$ is undefined. Using the above notation, Hamel and Goulden's theorem on the Giambelli-type formulas for the skew Schur function can be formulated as follows.

Theorem 4.1 ([6, Theorem 3.1]). *For an outside decomposition \mathbf{D} with k border strips B_1, B_2, \dots, B_k , we have*

$$s_{\lambda/\mu} = \det (s_{[\tau(\text{init}(B_i)), \tau(\text{fin}(B_j))])}_{i,j=1}^k. \quad (7)$$



a. A border strip decomposition b. An outside decomposition

Fig. 6. Border strip decompositions.

By choosing the outside decomposition whose border strips are the rows of the diagram of λ/μ in the above theorem, we obtain the Jacobi–Trudi identity for the skew Schur function, which states that

$$s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j})_{i,j=1}^{\ell(\lambda)}, \quad (8)$$

where h_k denotes the k th complete symmetric function, $h_0 = 1$ and $h_k = 0$ for $k < 0$.

Let $y(\lambda/\mu) = (t^{-\text{rank}(\lambda/\mu)} s_{\lambda/\mu}(1^t))_{t=0}$. The zrank conjecture says that $y(\lambda/\mu) \neq 0$ for any skew partition λ/μ . Now we give the evaluation of $y(\lambda/\mu)$ by using Theorem 4.1. First we consider the case when λ/μ is a border strip. In this case we have $\text{rank}(\lambda/\mu) = 1$, $\mu_i = \lambda_{i+1} - 1$ for $i \leq \ell(\lambda) - 1$ and $\mu_{\ell(\lambda)} = 0$. From the Jacobi–Trudi identity one easily deduces the following lemma.

Lemma 4.2. *For a border strip λ/μ we have*

$$y(\lambda/\mu) = \frac{(-1)^{\ell(\lambda)+1}}{\lambda_1 + \ell(\lambda) - 1}. \quad (9)$$

In order to compute $y(\lambda/\mu)$ for a general skew partition λ/μ , we need to consider the greedy border strip decomposition \mathbf{D}_0 of λ/μ . Suppose that $\text{rank}(\lambda/\mu) = r$. It follows that \mathbf{D}_0 has r border strips. We may apply Theorem 4.1 to \mathbf{D}_0 because it is also an outside decomposition. Furthermore, we may impose a canonical order on the strips B_1, B_2, \dots, B_r of \mathbf{D}_0 by the contents of their lower left-hand squares such that $\tau(\text{init}(B_i)) < \tau(\text{init}(B_{i+1}))$ for $i < r$. Since the sum of the heights of border strips in \mathbf{D}_0 is uniquely determined by the shape λ/μ , one sees that

$$z(\lambda/\mu) = ht(B_1) + ht(B_2) + \dots + ht(B_r)$$

is well defined. Let $\mathcal{J}_0 = \{(w_1, y_1), (w_2, y_2), \dots, (w_r, y_r)\}$ be the interval set of λ/μ with $\text{cr}(\mathcal{J}_0) = 0$. By Proposition 3.1 and the properties of \mathbf{D}_0 and \mathcal{J}_0 , we obtain that

$$\tau(\text{init}(B_i)) = \epsilon + w_i - 1 \quad \text{and} \quad \tau(\text{fin}(B_i)) = \epsilon + y_i - 2, \quad (10)$$

where ϵ is the smallest value among the contents of the squares of λ/μ .

The following theorem gives a determinantal formula for $y(\lambda/\mu)$ based on a matrix related to the Cauchy matrix.

Theorem 4.3. *Let λ/μ be a skew partition with $\text{rank}(\lambda/\mu) = r$, and let \mathcal{J}_0 be the noncrossing interval set $\{(w_1, y_1), (w_2, y_2), \dots, (w_r, y_r)\}$ of λ/μ . Then we have*

$$y(\lambda/\mu) = (-1)^{z(\lambda/\mu)} \det(d_{ij})_{i,j=1}^r, \quad (11)$$

where

$$d_{ij} = \begin{cases} \frac{1}{y_j - w_i}, & \text{if } y_j > w_i, \\ 0, & \text{if } y_j < w_i. \end{cases}$$

Proof. Take the greedy outside decomposition $\mathbf{D}_0 = \{B_1, B_2, \dots, B_r\}$ of λ/μ , and let ϕ_0 be the cutting strip corresponding to \mathbf{D}_0 . By Theorem 4.1 we have

$$s_{\lambda/\mu} = \det \left(s_{[\tau(\text{init}(B_i)), \tau(\text{fin}(B_j))]} \right)_{i,j=1}^r. \quad (12)$$

Suppose that the square with content $\tau(\text{init}(B_i))$ lies in the p_i th row of ϕ_0 , and the square with content $\tau(\text{fin}(B_j))$ lies in the q_j th row. Applying Lemma 4.2, we get

$$(t^{-1} s_{[\tau(\text{init}(B_i)), \tau(\text{fin}(B_j))]})_{t=0} = \frac{(-1)^{p_i - q_j}}{\tau(\text{fin}(B_j)) + 1 - \tau(\text{init}(B_i))} \quad (13)$$

if $[\tau(\text{init}(B_i)), \tau(\text{fin}(B_j))]$ is a substrip of ϕ_0 . Otherwise, the above entry is set 0. Note that $[\tau(\text{init}(B_i)), \tau(\text{fin}(B_j))]$ cannot be an empty strip for the greedy border strip decomposition. Using (10) we may write (13) as

$$(t^{-1} s_{[\tau(\text{init}(B_i)), \tau(\text{fin}(B_j))]})_{t=0} = \frac{(-1)^{p_i - q_j}}{y_j - w_i} \quad (14)$$

for $y_j > w_i$, or 0 for $y_j < w_i$. Thus, we have

$$y(\lambda/\mu) = (t^{-r} s_{\lambda/\mu}(1^t))_{t=0} = \det \left((t^{-1} s_{[\tau(\text{init}(B_i)), \tau(\text{fin}(B_j))]})_{t=0} \right)_{i,j=1}^r. \quad (15)$$

Extracting the signs from the determinant, we obtain

$$y(\lambda/\mu) = (-1)^{(p_1 + \dots + p_r) - (q_1 + \dots + q_r)} \det(d_{ij})_{i,j=1}^r = (-1)^{z(\lambda/\mu)} \det(d_{ij})_{i,j=1}^r.$$

This completes the proof. \square

Remark. Stanley [11] pointed out that one can also get a matrix for $y(\lambda/\mu)$ by taking the Jacobi–Trudi matrix (the matrix appearing in the Jacobi–Trudi determinant formula of $s_{\lambda/\mu}$) for the skew Schur function $s_{\lambda/\mu}$, and deleting all rows and columns that contain a 1, and then substituting $1/i$ for h_i . This matrix coincides with the matrix $(d_{ij})_{i,j=1}^r$ defined in (11), subject to permutations of rows and columns. This fact can be verified by using the transformation formula in [3].

From Theorem 4.3 and Proposition 2.1 one can recover the following expansion formula of Stanley [10, Eq. (30)].

Corollary 4.4. *We have*

$$y(\lambda/\mu) = (-1)^{z(\lambda/\mu)} \sum_{\mathcal{J}=\{(u_1, v_1), \dots, (u_r, v_r)\}} \frac{(-1)^{\text{cr}(\mathcal{J})}}{\prod_{i=1}^r (v_i - u_i)}, \quad (16)$$

summed over all interval sets \mathcal{J} of λ/μ .

5. An equivalent description of the zrank conjecture

We begin this section with the definition of a restricted Cauchy matrix. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two integer sequences. Suppose that a is strictly

decreasing and b is strictly increasing, and for any i, j we have $a_i > b_{n+1-i}$ and $a_i \neq b_j$. We define a matrix $C(a, b) = (c_{ij})_{i,j=1}^n$ by setting

$$c_{ij} = \begin{cases} \frac{1}{a_i - b_j}, & \text{if } a_i > b_j, \\ 0, & \text{if } a_i < b_j. \end{cases}$$

Definition 5.1. A matrix M is called a *restricted Cauchy matrix* if there exist two integer sequences a and b satisfying the above conditions such that $M = C(a, b)$.

For a matrix M we say it is *singular* if $\det(M) = 0$; or *nonsingular*, otherwise. We now come to the main result of this paper.

Theorem 5.2. *The following two statements are equivalent:*

- (i) *The zrank conjecture is true for any skew partition.*
- (ii) *Any restricted Cauchy matrix is nonsingular.*

Proof. Suppose that (ii) is true. For a skew partition λ/μ , consider the noncrossing interval set $\mathcal{I}_0 = \{(w_1, y_1), (w_2, y_2), \dots, (w_r, y_r)\}$ of λ/μ . Clearly, $w_i \neq y_j$ for $1 \leq i, j \leq r$. Let $w = (w'_1, w'_2, \dots, w'_r)$ be the rearrangement of (w_1, w_2, \dots, w_r) in increasing order, and let $y = (y'_1, y'_2, \dots, y'_r)$ be the rearrangement of (y_1, y_2, \dots, y_r) in decreasing order. For $1 \leq i \leq r$, we have $y'_i > w'_{r+1-i}$ since the number of $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$ columns in the first ℓ columns of the reduced code $c(\lambda/\mu)$ is bigger than or equals to the number of $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$ columns for $1 \leq \ell \leq k$, where k is the length of $c(\lambda/\mu)$. Notice that the determinant $\det(d_{ij})_{i,j=1}^r$ appearing in (11) is equal to the determinant of the restricted Cauchy matrix $C(y, w)$ up to a sign. By Theorem 4.3 we see that

$$y(\lambda/\mu) \neq 0 \Leftrightarrow \det(d_{ij})_{i,j=1}^r \neq 0 \Leftrightarrow \det(C(y, w)) \neq 0.$$

Since the matrix $C(y, w)$ is nonsingular, we have $\text{rank}(\lambda/\mu) = \text{zrank}(\lambda/\mu)$.

Now we proceed to prove (ii) by assuming that (i) is true. Given a restricted Cauchy matrix $C(a, b)$ of order r , without loss of generality, we may assume that a and b are sequences of positive integers. Let λ be the partition with $\lambda_i = a_i - r + i$, and let μ be the partition with $\mu_i = b_{r+1-i} - r + i$. From $a_i > b_{r+1-i}$ we may deduce $\lambda_i > \mu_i$ for all i . Thus we can construct a skew diagram λ/μ . Observe that the Jacobi–Trudi matrix $(h_{\lambda_i - \mu_j - i + j})$ of $s_{\lambda/\mu}$ does not have a column containing 1 since

$$\lambda_i - \mu_j - i + j = a_i - b_{r+1-j} \neq 0, \quad \text{for } 1 \leq i, j \leq r.$$

It follows that $\text{rank}(\lambda/\mu) = r$ from [10, Proposition]. Therefore, we have

$$y(\lambda/\mu) = (t^{-r} s_{\lambda/\mu}(1^t))_{t=0} = \det \left((t^{-1} h_{\lambda_i - \mu_j - i + j}(1^t))_{t=0} \right)_{i,j=1}^r,$$

which is the determinant $\det(C(a, b))$ up to a sign. If the zrank conjecture is true for λ/μ , then we have $y(\lambda/\mu) \neq 0$, implying that $C(a, b)$ is nonsingular. This completes the proof. \square

We remark that we may restrict our attention to irreducible restricted Cauchy matrices for the verification of the zrank conjecture. In other words, if every irreducible restricted Cauchy matrix is nonsingular, then every restricted Cauchy matrix is nonsingular.

6. Special cases

In this section we consider several classes of restricted Cauchy matrices $C(a, b) = (c_{ij})_{i,j=1}^r$ for which we can prove that they are nonsingular.

Class I. For all i, j we have $r_{ij} \neq 0$.

In this case, $(c_{ij})_{i,j=1}^r$ is a Cauchy matrix. Cauchy [7] showed that

$$\det \left(\frac{1}{a_i - b_j} \right)_{i,j=1}^r = \prod_{i < j} (a_i - a_j) \prod_{i < j} (b_j - b_i) \prod_{i,j} \frac{1}{a_i - b_j}. \quad (17)$$

It follows that

$$\det(c_{ij})_{i,j=1}^r > 0.$$

From the proof of [10, Theorem 3.2 (b)], we get

Proposition 6.1. For a connected skew diagram λ/μ , if every row of the Jacobi–Trudi matrix that contains a 0 also contains a 1, then the matrix $(d_{ij})_{i,j=1}^r$ appearing in (11) must satisfy that $d_{ij} \neq 0$ for all i, j .

Theorem 4.3 and Proposition 6.1 yield another proof of [10, Theorem 3.2] of Stanley. Some skew partitions do not have the property stated in the above proposition, but the matrices $(d_{ij})_{i,j=1}^r$ are Cauchy matrices. For instance, taking $\lambda/\mu = (8, 8, 7, 7, 7, 6, 1)/(5, 5, 3, 3, 2)$, its Jacobi–Trudi matrix is

$$s_{(8,8,7,7,7,6,1)/(5,5,3,3,2)} = \begin{vmatrix} h_3 & h_4 & h_7 & h_8 & h_{10} & h_{13} & h_{14} \\ h_2 & h_3 & h_6 & h_7 & h_9 & h_{12} & h_{13} \\ 1 & h_1 & h_4 & h_5 & h_7 & h_{10} & h_{11} \\ 0 & 1 & h_3 & h_4 & h_6 & h_9 & h_{10} \\ 0 & 0 & h_2 & h_3 & h_5 & h_8 & h_9 \\ 0 & 0 & 1 & h_1 & h_3 & h_6 & h_7 \\ 0 & 0 & 0 & 0 & 0 & 1 & h_1 \end{vmatrix}.$$

Class II. For all $(i, j) \neq (r, r)$, we have $c_{ij} \neq 0$ and $c_{rr} = 0$.

Let

$$M = \prod_{i=1}^{r-1} \frac{(a_r - a_i)(b_i - b_r)}{(a_r - b_i)(a_i - b_r)}.$$

Since $b_r > a_r$, it is easy to show that $M > 1$. We see that the restricted Cauchy matrix in this case is of the following form:

$$(c_{ij})_{i,j=1}^r = \begin{pmatrix} \frac{1}{a_1-b_1} & \cdots & \frac{1}{a_1-b_{r-1}} & \frac{1}{a_1-b_r} \\ \frac{1}{a_2-b_1} & \cdots & \frac{1}{a_2-b_{r-1}} & \frac{1}{a_2-b_r} \\ \vdots & \cdots & \vdots & \vdots \\ \frac{1}{a_{r-1}-b_1} & \cdots & \frac{1}{a_{r-1}-b_{r-1}} & \frac{1}{a_{r-1}-b_r} \\ \frac{1}{a_r-b_1} & \cdots & \frac{1}{a_r-b_{r-1}} & 0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \det(c_{ij})_{i,j=1}^r &= \prod_{\substack{i,j=1 \\ i < j}}^r (a_i - a_j)(b_j - b_i) \prod_{i,j=1}^r \frac{1}{a_i - b_j} \\ &\quad - \frac{1}{a_r - b_r} \prod_{\substack{i,j=1 \\ i < j}}^{r-1} (a_i - a_j)(b_j - b_i) \prod_{i,j=1}^{r-1} \frac{1}{a_i - b_j} \\ &= \frac{1}{a_r - b_r} \prod_{\substack{i,j=1 \\ i < j}}^{r-1} (a_i - a_j)(b_j - b_i) \prod_{i,j=1}^{r-1} \frac{1}{a_i - b_j} (M - 1). \end{aligned}$$

It follows that

$$\det(c_{ij})_{i,j=1}^r < 0.$$

Class III. $c_{ij} \neq 0$ except for c_{rr} , $c_{r,r-1}$ and $c_{r-1,r}$.

In this case, we have $a_r > b_{r-2}$ but $a_r < b_{r-1}$, $a_{r-2} > b_r$ but $a_{r-1} < b_r$. Recall that the *rank* of a matrix is the maximum number of linearly independent rows or columns of the matrix. For a matrix $M = (m_{ij})_{i,j=1}^r$, let M^* be the matrix $(M_{ji})_{i,j=1}^r$, where M_{ij} is the cofactor of m_{ij} in the expansion $\det(M) = \sum_{i=1}^r m_{ij} M_{ij}$. Recall the following property:

$$\text{rank}(M^*) = \begin{cases} r, & \text{if } \text{rank}(M) = r, \\ 1, & \text{if } \text{rank}(M) = r - 1, \\ 0, & \text{if } \text{rank}(M) < r - 1. \end{cases} \quad (18)$$

We now consider the rank of $C^* = (C_{ji})_{i,j=1}^r$ where C_{ij} is the cofactor of m_{ij} in the expansion $\det(C(a, b)) = \sum_{i=1}^r c_{ij} C_{ij}$. Recall that the minor C_{rr} is the determinant of the submatrix obtained from $C(a, b)$ by deleting row r and column r , which

turns out to be the restricted Cauchy matrix of Class I, and the underlying matrices of $C_{r-1,r-1}$, $C_{r,r-1}$, $C_{r-1,r}$ are the restricted Cauchy matrices of Class II. Thus we have

$$C_{r,r} > 0, \quad C_{r-1,r-1} < 0, \quad C_{r,r-1} > 0 \quad \text{and} \quad C_{r-1,r} > 0.$$

This implies that $\text{rank}(C^*) \geq 2$. Hence $\text{rank}(C(a, b)) = r$ because of (18), namely $\det(c_{ij})_{i,j=1}^r \neq 0$.

Class IV. For all $i \leq r$, $j \leq r-1$ we have $c_{ij} \neq 0$; $c_{1r} \neq 0$, and $c_{2r} \neq 0$; $c_{ir} = 0$ if $i > 2$.

In this case, the restricted Cauchy matrix has the form

$$(c_{ij})_{i,j=1}^r = \begin{pmatrix} \frac{1}{a_1-b_1} & \cdots & \frac{1}{a_1-b_{r-1}} & \frac{1}{a_1-b_r} \\ \frac{1}{a_2-b_1} & \cdots & \frac{1}{a_2-b_{r-1}} & \frac{1}{a_2-b_r} \\ \frac{1}{a_3-b_1} & \cdots & \frac{1}{a_3-b_{r-1}} & 0 \\ \vdots & \cdots & \vdots & \vdots \\ \frac{1}{a_r-b_1} & \cdots & \frac{1}{a_r-b_{r-1}} & 0 \end{pmatrix}.$$

Expanding along the last column, we get

$$\begin{aligned} \det(c_{ij})_{i,j=1}^r &= (-1)^{r+1} \frac{1}{a_1-b_r} \prod_{2 \leq i < j \leq r} (a_i - a_j) \prod_{1 \leq i < j \leq r-1} (b_j - b_i) \prod_{\substack{2 \leq i \leq r \\ 1 \leq j \leq r-1}} \frac{1}{a_i - b_j} \\ &\quad + (-1)^{r+2} \frac{1}{a_2-b_r} \prod_{\substack{i \neq 2, j \neq 2 \\ 1 \leq i < j \leq r}} (a_i - a_j) \prod_{1 \leq i < j \leq r-1} (b_j - b_i) \prod_{\substack{i \neq 2 \\ 1 \leq j \leq r-1}} \frac{1}{a_i - b_j} \\ &= (-1)^{r+1} \prod_{1 \leq i < j \leq r} (a_i - a_j) \prod_{1 \leq i < j \leq r-1} (b_j - b_i) \prod_{\substack{i,j=1 \\ j \neq r}}^r \frac{1}{a_i - b_j} N, \end{aligned}$$

where

$$N = \frac{f(a_1) - f(a_2)}{a_1 - a_2}$$

and

$$f(x) = \frac{(x-b_1)(x-b_2) \cdots (x-b_{r-1})}{(x-b_r)(x-a_3) \cdots (x-a_r)}.$$

Let $\delta = a_1 - a_2$. We obtain

$$\begin{aligned} \frac{f(a_1)}{f(a_2)} &= \frac{\frac{(a_1-b_1)(a_1-b_2) \cdots (a_1-b_{r-1})}{(a_1-b_r)(a_1-a_3) \cdots (a_1-a_r)}}{\frac{(a_2-b_1)(a_2-b_2) \cdots (a_2-b_{r-1})}{(a_2-b_r)(a_2-a_3) \cdots (a_2-a_r)}} \\ &= \frac{\frac{(a_1-b_1)}{(a_2-b_1)} \frac{(a_1-b_2)}{(a_2-b_2)} \cdots \frac{(a_1-b_{r-1})}{(a_2-b_{r-1})}}{\frac{(a_1-b_r)}{(a_2-b_r)} \frac{(a_1-a_3)}{(a_2-a_3)} \cdots \frac{(a_1-a_r)}{(a_2-a_r)}} \end{aligned}$$

$$= \frac{\frac{(\delta+a_2-b_1)}{(a_2-b_1)} \frac{(\delta+a_2-b_2)}{(a_2-b_2)} \cdots \frac{(\delta+a_2-b_{r-1})}{(a_2-b_{r-1})}}{\frac{(\delta+a_2-b_r)}{(a_2-b_r)} \frac{(\delta+a_2-a_3)}{(a_2-a_3)} \cdots \frac{(\delta+a_2-a_r)}{(a_2-a_r)}}.$$

Let $s \in \{b_1, \dots, b_{r-1}\}$ and $s' \in \{a_3, \dots, a_r, b_r\}$. Then we have $s < s'$ and

$$\frac{(\delta + a_2 - s)}{(a_2 - s)} < \frac{(\delta + a_2 - s')}{(a_2 - s')}. \quad (19)$$

It follows that $f(a_1) < f(a_2)$, namely $N < 0$. Thus we have $\det(c_{ij})_{i,j=1}^r > 0$ if r is even and $\det(c_{ij})_{i,j=1}^r < 0$ if r is odd.

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